On Optimal Paths in Free Spaces Including Obstacles with Fuzzy Boundaries of Fuzzy Normed Spaces

Seiji Saito*
Hiroaki Ishii**
Kuang-Yih Yeh***
Hao-Ching Hsia***

Abstract

In this paper we introduce definitions of distances defined by norms in linear spaces in order to find optimal paths in free spaces with obstacles. Secondly, Euler-Lagrange equations in the calculus of variation give the optimal solutions for the problems of optimal paths. Thirdly we consider the linear structure in a sets of fuzzy numbers and also introduce norms in f fuzzy linear spaces. Finally we discuss optimal, i.e., shortest paths in free spaces including obstacles, with fuzzy boundaries, because it is useful in getting shortest paths in realistic environment with natural damage, for example, earthquakes etc.

1. Norm and Gauge

Location problems with barriers is interesting from a mathematical point of view [K]. The nonconvexity of distance measures in the presence of barriers leads to nonconvex optimization problems. Minkowski defined a norm and showed proper-

*Faculty of Engineering, Doshisha University
**Graduate School of Information Science and Technology, Osaka University
***Urban Planning Department, National Cheng Kung University

ties of the norme (see [K]):

Definition M1 (Minkowski 1911) Let S be a compact convex subset in $\mathbf{R}^n$ containing the origin in its interior. Let S be symmetric with respect to the origin and let x be in $\mathbf{R}^n$. The norm $\gamma:\mathbf{R}^n \to \mathbf{R}$ of x with respect to S is defined as

$$\gamma(x)=\inf\{c>0: x\in cS\}.$$

We will often refer $\gamma(x)$ to $||x||$.

Lemma ML1 (Minkowski 1911) Let γ be defined corresponding to the above definition. Then x, $y\in\mathbf{R}^n$ and $c\in\mathbf{R}$, the following properties (i)-(iii) hold;

(i) $\gamma(x)\geqq 0$ and $[\gamma(x)=0 \Leftrightarrow x=0]$;

(ii) $\gamma(cx)=|c|\gamma(x)$;

(iii) $\gamma(x+y)\leqq\gamma(x)+\gamma(y)$.

Every norm γ defines a distance measure in $\mathbf{R}^n$ in the following way:

$$\gamma(x, y)=\gamma(x-y)=||x-y||_\gamma.$$

We will write $d(x, y)$ instead of $\gamma(x, y)$ and refer to the norm with the introduced metric d as $||\cdot||_d$.

If the symmetry assumptions is dropped in the definition of s distance measure, we obtain the more general concept of **gauge**.

Definition M2 (Minkowski 1911) Let S be a compact convex subset in $\mathbf{R}^n$ containing the origin in its interior and let x be in $\mathbf{R}^n$. The gauge $\gamma:\mathbf{R}^n \to \mathbf{R}$ of x with respect to S is defined as

$$\gamma(x)=\inf\{c>0: x\in cS\}.$$

The following properties hold.

Lemma ML2 (Minkowski 1911) Let γ be defined corresponding to the above Definition D2. Then x, $y\in\mathbf{R}^n$ and $c\in\mathbf{R}$, the following properties (i)-(iii) hold;

(i) $\gamma(x)\geqq 0$ and $[\gamma(x)=0 \Leftrightarrow x=0]$;

(ii) $\gamma(cx)=c\gamma(x)$ $(c\geqq 0)$;

(iii) $\gamma(x+y)\leqq\gamma(x)+\gamma(y)$.

The above gauge is playing an important role in finding shortest paths in a space with obstacles.

2. Euler-Lagrange Equation

Let a prescribed metric d be given in the n-dimensional space $\mathbf{R}^n$. We assume that the metric d is introduced by a norm $\|\cdot\|_d : \mathbf{R}^n \to \mathbf{R}$ as $d(x, y) = \|x - y\|_d$. Let $\{B_1, B_2, \cdots, B_N\}$ be a finite set of closed and pairwise disjoint sets in $\mathbf{R}^n$ with nonempty interior. Each set B_i, $i = 1, 2, \cdots, N$, is called a barrier (or obstacle) and the union $B = \cup_{i=1}^{N} B_i$. Let $F = \mathbf{R}^n\text{-int}(B)$ be the feasible region (free space).

Definition P. Let x, y be in the free space F. A continuous curve P given by the parameterization $p = (p_1, p_2, \cdots, p_n)^T$, where T means the transpose, is a function defined on the interval $[0, 1]$ to $\mathbf{R}^n$ with $p(0) = x$, $p(1) = y$ that is continuous differentiable on $[0, 1]$ with the possible exception of at most a finite number of points, where the derivative p' has finite limits from the left and the from the right, is called an **x-y path.** The length $L(P)$ of the x-y path P with respect to the prescribed metric d is given by

$$L(P) = \int_0^1 \|p'(t)\|_d dt.$$

If P does not intersect the interior of a barrier, i.e., $p([0, 1]) \cap \text{int}(B) = \phi$, the x-y path P is called a **permitted x-y path.** The shortest path metric between two points x, y in F is denoted by $d_B(x, y) = \inf\{L(P)$: P permitted x-y path$\}$, which is called a **d-shortest permitted x-y path.**

For simplicity we assume that the following conditions:

(1) B is one compact and convex barrier;

(2) the boundary ∂B is (n-1)-dimensional smooth manifold in $\mathbf{R}^n$ with $\partial B = \{x \text{ in } \mathbf{R}^n : G(x) = 0\}$, where G is twice continuous differentiable and $\nabla G(x) \neq 0$.

Under the above assumptions, the problem of finding a d-shortest permitted x-y path is equivalent to the following minimization problem over all feasible and piecewise continuous differentiable x-y paths P with parametrization p.

$$\min \int_0^1 F(t, p(t), p'(t))dt, \quad F(t, p(t), p') = \|dp/dt\|_d$$

$$s.t.\ p(0) = x,\ p(1) = y,$$

$$p(t) \notin \mathrm{int}(B) \quad t \in [0, 1].$$

In an example[K] in which G(x)=0 is unit circle we get the condition that a d-shortest permitted x-y path P satisfies the following Euler-Lagrange equations.

We assume that $G(x) = \sqrt{\sum_{i=1}^{n} x_i^2} - 1$.

Theorem K. (Euler-Lagrange equation) A d-shortest permitted x-y path P satisfies the following equations (a), or, (b) ;

(a) p satisfies the following Euler-Lagrange equations:

$$\frac{\partial F}{\partial p_j}(t, p, p') - \frac{d}{dt}\frac{\partial F}{\partial q_j}(t, p, p') = 0 \quad (j = 1, 2, \cdots, n),\ q = p';$$

(b) p satisfies G(p)=0 and the following equations:

$$\frac{\partial F}{\partial p_i}(t, p, p') - \frac{d}{dt}\frac{\partial F}{\partial q_i}(t, p, p') + \lambda(t)\frac{\partial G}{\partial p_i}(p) = 0 \quad (i = 1, 2, \cdots, n),\ q = p'.$$

3. Fuzzy Normed Space

In this section we introduce a linear structure into sets of fuzzy numbers and define a norm in the fuzzy linear space. Denote I=[0, 1]. The following definition means that a fuzzy number can be identified with a membership function.

Definition 1. Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

$F_b^{st} = \{\mu : \mathbf{R} \to I$ satisfies (i)−(iv) below.$\}$

(i) μ has a unique number $m \in \mathbf{R}$ such that $\mu(m) = 1$ (normality);

(ii) $\mathrm{supp}(\mu) = \mathrm{cl}(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ is bounded $\in \mathbf{R}$ (bounded support);

(iii) μ is strictly fuzzy convex on supp(μ) as follows:

(a) if supp(μ) $\neq$ {m}, then $\mu(c\xi_1+(1-c)\xi_2)>\min[\mu(\xi_1), \mu(\xi_2)]$ for, ξ_1, $\xi_2 \in$ supp(μ) with $\xi_1 \neq \xi_2$ and $0<c<1$;

(b) if supp(μ) $=$ {m}, then $\mu(\mathrm{m})=1$ and $\mu(\xi)=0$ for $\xi \neq \mathrm{m}$;

(iv) μ is upper semi-continuous on **R**.

It follows that $\mathbf{R} \subset F^{st}_{b}$. Because m has a membership function as follows:

$\mu(\mathrm{m})=1$; $\mu(\xi)=0$ for $\xi \neq \mathrm{m}$.

Then μ satisfies the above (i)-(iv).

In usual case a fuzzy number x satisfies **fuzzy convex** on **R**, i.e.,

(3.1) $$\mu(c\xi_1+(1-c)\xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)]$$

for $0 \leq c \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$.

Denote α-cut sets by $\mathrm{L}_\alpha(\mu)=\{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$ for $\alpha \in \mathrm{I}$. When the membership function is fuzzy convex, then we have the following remarks.

Remark 1. The following statements (1)-(4) are equivalent each other, provided with (i) of Definition 1.

(1) (3.1) holds;

(2) $\mathrm{L}_\alpha(\mu)$ is convex with respect to $\alpha \in \mathrm{I}$;

(3) μ is non-decreasing in $\xi \in (-\infty, \mathrm{m})$, non-increasing in $\xi \in (\mathrm{m}, +\infty)$, respectively;

(4) $\mathrm{L}_\alpha(\mu) \subset \mathrm{L}_\beta(\mu)$ for $\alpha > \beta$.

The above condition (iiia) is stronger than (3.1). From (iiia) it follows that $\mu(\xi)$ is strictly monotonously increasing in $\xi \in [\min \mathrm{supp}(\mu), \mathrm{m}]$. Suppose that $\mu(\xi_1) \geq \mu(\xi_2)$ for $\xi_1 < \xi_2 \leq \mathrm{m}$. From Remark 1(3), it follows that $\mu(\xi_1)=\mu_1(\xi_2)$ for some $\xi_1 < \xi_2$, so we get $\mu(\xi)=\mu(\xi_1)=\mu(\xi_2)$ for $\xi \in [\xi_1, \xi_2]$. This contradicts with Definition 1 (iiia). Thus μ is strictly monotonously increasing. In the similar way

μ is strictly monotonously decreasing in $\xi \in [m, \max \mathrm{supp}(\mu)]$. This condition plays an important role in Theorem 1.

We introduce the following parametric representation of $\mu \in F_b^{st}$ as

$$x_1(\alpha) = \min L_\alpha(\mu), \quad x_2(\alpha) = \max L_\alpha(\mu) \quad (0 < \alpha \leqq 1)$$

and

$$x_1(0) = \min \mathrm{supp}(\mu), \quad x_2(0) = \max \mathrm{supp}(\mu).$$

In what follows we denote $\mu = (x_1, x_2)$.

Denote by $C(I)$ the set of all the continuous functions on I to **R**. The following theorem shows that a membership function is characterized by x_1, x_2.

Theorem 1. Denote the left-, right-end points of the α-cut set of $\mu \in F_b^{st}$ by $x_1(\alpha)$, $x_2(\alpha)$, respectively. Here $x_1(\alpha)$ and $x_2(\alpha)$ are **R**-valued functions defined on I. The following properties (i)-(iii) hold.

(i) $x_1, x_2 \in C(I)$;

(ii) $\max\{x_1(\alpha): \alpha \in I\} = x_1(1) = m = \min\{x_1(\alpha): \alpha \in I\} = x_2(1)$;

(iii) x_1, x_2 are non-decreasing, non-increasing on I, respectively and the one of the following statements (a) and (b) holds:

(a) there exists a positive number $c \leqq 1$ such that

$x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c]$

and that

$x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$.

Conversely, under the above conditions (i)-(iii), by denoting

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leqq \xi \leqq x_2(\alpha)\}$$

for $\xi \in \mathbf{R}$, then $\mu \in F_b^{st}$.

The proof is omitted.

Remark 2. From the above Condition (i) a fuzzy number $x=(x_1, x_2)$ means a bounded continuous curve over $\mathbf{R}^2$ and $x_1(\alpha)\leqq x_2(\alpha)$ for $\alpha\in I$.

In what follows we denote $\mu=(x_1, x_2)$ for $\mu\in F_b^{st}$. The parametric representation of $\mu\in F_b^{st}$ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g: \mathbf{R}\times\mathbf{R}\rightarrow\mathbf{R}$ be a function. The corresponding binary operation of two fuzzy numbers $x, y\in F_b^{st}$ to $g(x, y): F_b^{st}\times F_b^{st}\rightarrow F_b^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x, y)}$ of g is as follows:

$$\mu_{g(x, y)}(\xi)=\sup\{\min[\mu_x(\xi_1), \mu_y(\xi_2)]: \xi=g(\xi_1, \xi_2)\}.$$

Here $\xi, \xi_1, \xi_2\in\mathbf{R}$ and μ_x, μ_y are membership functions of x, y, respectively. From the extension principle, it follows that, in case where $g(x, y)=x+y$,

$$\begin{aligned}&\mu_{x+y}(\xi)\\ &=\max\{\min_{i=1,2}\mu_i(\xi_i): \xi=\xi_1+\xi_2\} \text{ where } \mu_1=\mu_x, \mu_2=\mu_y\\ &=\max\{\alpha\in I: \xi=\xi_1+\xi_2, \xi_i \text{ in } L_\alpha(\mu_i) \text{ for } i=1, 2\}\\ &=\max\{\alpha\in I: \xi \text{ in}[x_1(\alpha)+y_1(\alpha), x_2(\alpha)+y_2(\alpha)]\}.\end{aligned}$$

Thus we get $x+y=(x_1+y_1, x_2+y_2)$. In the similar way $x-y=(x_1-y_2, x_2-y_1)$.
Denote a metric by

$$d_\infty(x, y)=\sup_{\alpha\in I}\max(|x_1(\alpha)-y_1(\alpha)|, |x_2(\alpha)-y_2(\alpha)|)$$

for $x=(x_1, x_2), y=(y_1, y_2)\in F_b^{st}$.

Theorem 2. F_b^{st} is a complete metric space in $C(I)^2$.
The proof is omitted.

In what follows we introduce a linear structure into the set of fuzzy numbers and a norm over the fuzzy linear space. According to the extension principle of Zadeh, for respective membership functions μ_x, μ_y of $x, y\in F_b^{st}$ and $c\in\mathbf{R}$, the following addition and scalar product are given as follows:

Addition: $\mu_{x+y}(\xi)=\sup\{\alpha\in[0, 1]: \xi=\xi_1+\xi_2$ for ξ_1 in $L_\alpha(\mu_x)$ and ξ_2 in $L_\alpha(\mu_y)\}$;

Scalar product: $\mu_{cx}(\xi)=\mu_x(\xi/c)$ for $c\neq 0$,

$=0$ for $c=0$ and $\xi\neq 0$,

$=1$ for $c=0$ and $\xi=0$.

In [PR] they introduced an equivalence relation $(x, y)\sim(u, v)$ for (x, y), $(u, v)\in F_b^{st}\times F_b^{st}$, i.e.,

$$(x, y)\sim(u, v) \Leftrightarrow x+v=u+y.$$

Putting $x=(x_1, x_2)$, $y=(y_1, y_2)$, $u=(u_1, u_2)$, $v=(v_1, v_2)$ by the parametric representation, the above equivalence relation means that the following equations hold:

$$x_i+v_i=u_i+y_i \quad (i=1, 2)$$

Denote an equivalence class by

$$[x, y]=\{(u, v)\in F_b^{st}\times F_b^{st}: (u, v)\sim(x, y)\} \text{ for } x, y\in F_b^{st}$$

and the set of equivalence classes by

$$F_b^{st}\times F_b^{st}/\sim=\{[x, y]: x, y\in F_b^{st}\}$$

such that the one of following cases (i) and (ii) holds:

(i) if $(x, y)\sim(u, v)$, then $[x, y]=[u, v]$;

(ii) if not $(x, y)\sim(u, v)$, then $[x, y]\cap[u, v]=\phi$.

Then $F_b^{st}\times F_b^{st}/\sim$ is a linear space with the following addition and scalar product

$$[x, y]+[u, v]=[x+u, y+v];$$

$$c[x, y]=[cx, cy] \quad \text{for } c\geqq 0,$$

$$=[(-c)y, (-c)x] \quad \text{for } c<0,$$

for $c\in\mathbf{R}$ and $[x, y]$, $[u, v]\in F_b^{st}\times F_b^{st}/\sim$. They denote a norm $\in F_b^{st}\times F_b^{st}/\sim$ by

$$\|[x, y]\|=\sup_{\alpha\in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here d_H is the Hausdorff metric between compact subsets $L_\alpha(\mu_x)$ and $L_\alpha(\mu_y)$ is as follows:

$$d_H(L_\alpha(\mu_x), L_\alpha(\mu_x)) = \max(\sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|,$$
$$\sup_{\eta \in L_\alpha(\mu_x)} \inf_{\xi \in L_\alpha(\mu_y)} |\xi - \eta|).$$

It can be easily seen that $\|[x, y]\| = d_\infty(x, y)$.

Note that $\|[x, y]\| = 0 \in F_b^{st} \times F_b^{st} / \sim$ if and only if $x = y \in F_b^{st}$.

4. Discussion : Shortest Path in Free Space in Fuzzy Normed Space

In this section we discuss shortest paths in free spaces including obstacles with fuzzy boundaries of fuzzy normed spaces. In case where finite many obstacles in a free space we can find a shortest path permitted by applying Kalmroth's theorem in Section 2. In case where obstacles has uncertain boundaries, for example, after an earthquake happens in the free space including obstacles, their boundaries may be destroyed and we will have fuzzy information of the boundaries. Uncertain information of boundaries may be considered as fuzzy. In order to apply optimization theory and nonlinear functional analysis it is need to consider to optimization problems of finding shortest permitted paths in free spaces including obstacles with fuzzy boundaries as fuzzy optimization problems in fuzzy normed spaces of Section 3.

In the future studying we are proposing an algorithm for fuzzy optimization problems of finding shortest permitted paths in free space including obstacles with fuzzy boundaries by applying the calculus of variation to fuzzy normed space.

References

[K] K. Kalmroth: Single-Facilities Location Problems with Barriers, Springer, 2002.

[PR] M. L. Puri and D. A. Ralescu : Differential of Fuzzy Functions,J. Math. Anal. Appl. 91 (1983), 552-558.