On number of occurrences of runs in a higher order Markov tree

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Abstract.

In this paper, a higher order Markov property on a directed tree is studied. Recursive formulae of the conditional probability generating functions for deriving the exact distributions of numbers of non-overlapping "1"-runs of length k on a higher order Markov tree are obtained. Furthermore, we explain how to calculate the probability functions on higher order Markov tree with illustrative examples.

Key words and phrases: conditional independence, conditional probability generating function, higher order Markov tree, non-overlapping counting, probability generating function, runs.

1. Introduction

Recently, Aki (1999) derived the exact distributions of the number of non-overlapping "1"-runs of length k on a directed tree, whose vertices are assumed to be $\{0,1\}$-valued random variables which follow a directed Markov distribution (Lauritzen (1996), p. 52).

In this paper, we introduce a higher order Markov tree to model a complex dependent structure on a directed tree.

Let T be a directed tree and let V be the set of the vertices of T. We assume that $\{X_v, v \in V\}$ is a collection of $\{0, 1\}$-valued random variables indexed by the vertices of T. Then, on the assumption that the collection of $\{0, 1\}$-valued random variables on the directed tree is a higher order Markov tree, we derive the exact distributions of numbers of "1"-runs of a specified length.

To solve the problems, we adopt the method of conditional probability generating function (*pgf*)'s. Since the method of conditional *pgf*'s was introduced by Ebneshahrashoob and Sobel (1990), many researchers utilized the method to solve various problems (see Aki (1992), Uchida and Aki (1995), Aki *et al.* (1996), Balakrishnan *et al.* (1997), Han and Aki (2000) and others).

We give in Section 2 the definition of a homogeneous higher order Markov tree. A recursive formula for deriving the exact distribution of the number of non-overlapping "1"-runs of length k along the direction is obtained on the higher order Markov tree. By using the systems of equations of the conditional *pgf*'s recursively, we can obtain the *pgf*'s of the distribution of the number of non-overlapping "1"-runs of length k on the higher order Markov tree. Further, we illustrate in detail how to derive the *pgf*'s of the distributions of the numbers of non-overlapping "1"-runs of length k on the higher order Markov tree with an example.

In Section 3, we illustrate how to compute the probability function of the distribution of the number of non-overlapping "1"-runs of length k from the recursive formulae obtained in Sections 2. It is very easy to obtain the probabilities from the recursive formulae by using some computer algebra systems. As an illustrative example, we compute the probability function of the number of non-overlapping "1"-runs of a specified length when a directed tree is (i) independently and identically distributed (*iid*), (ii) a first order Markov tree, (iii) a second order Markov tree, respectively.

2. Number of non-overlapping "1"-runs of length k on higher order Markov tree

Let T be a directed tree and let V be the set of the vertices of T. The vertices are assumed to be labeled in increasing order away from the root of the tree so that a vertex is preceded by its ancestors. We denote by $v(n)$ the vertex labeled n and denote by $N(v)$ the number of a vertex v. Let $v(1)$ be the root of T and pa: $V\backslash\{v(1)\}\rightarrow V$ be the map from every vertex v to its parent. We fix any vertex v except for the root. Suppose that the vertex v has $a(v)$ ancestors, $v^1, v^2, \cdots, v^{a(v)}(=v(1))$ with $pa(v^j)=v^{j+1}$ for $j=1, \cdots, a(v)-1$. We define that $a(v(1))=0$. Then, the map a: $V\rightarrow \mathbf{Z}^+$ from every vertex v to the number of ancestors of v is defined. We assume that $\{X_v, v\in V\}$ is a collection of $\{0,1\}$-valued random variables and $n_T\equiv|V|$, $g_T\equiv\max\{a(v);\ v\in V\}$ (the maximum length of the paths in T). Here, for the basic notations in graphical models, see Lauritzen (1996), Ripley (1996) and Aki (1999).

Let E_m be the totality of $\{0,1\}$-sequences of length k less than or equal to m.

Definition 2.1. (the m-th order Markov tree) Let m be a positive integer less than g_T. $\{X_v, v\in V\}$ is a homogeneous m-th order Markov tree if it satisfies the following conditions:

(i) there exists $0<p<1$ such that $P(X_{v(1)}=1)=p$;

(ii) for every $\{0,1\}$-sequence $\mathbf{e}\in E_m$, there exists a real number $0<P(\mathbf{e})<1$

which satisfies the following condition:

for every $n=2, \cdots, n_T$, if $a(v(n))=j<m$, then

$$P(X_{v(n)}=1 \mid X_{v(n-1)}, \cdots, X_{v(1)}=e_1)$$
$$=P(X_{v(n)}=1 \mid X_{pa(v(n))}=e_{N(pa(v(n)))}, \cdots, X_{pa^j(v(n))}=e_{N(pa^j(v(n)))}(=e_1))$$
$$=p((e_{N(pa^j(v(n)))}, \cdots, e_{N(pa(v(n)))}));$$

for every $n=2, \cdots, n_T$, if $a(v(n))=j \geq m$, then

$$P(X_{v(n)}=1 \mid X_{v(n-1)}=e_{n-1}, \cdots, X_{v(1)}=e_1$$
$$=P(X_{v(n)}=1 \mid X_{pa(v(n))}=e_{N(pa(v(n)))}, \cdots, X_{pa^m(v(n))}=e_{N(pa^m(v(n)))})$$
$$=p((e_{N(pa^m(v(n)))}, \cdots, e_{N(pa(v(n)))})),$$

where pa^j means the j-th composition of the map pa.

For every vertex v, we denote by $c(v)$ the number of children of v and if $c(v)>0$, we denote the children of v by $v_1, v_2, \cdots, v_{c(v)}$. Let T_v be the (directed) subtree which consists of the vertex v (the root of the subtree) and of all of the descendants of v. V_v denotes the set of the vertices of T_v.

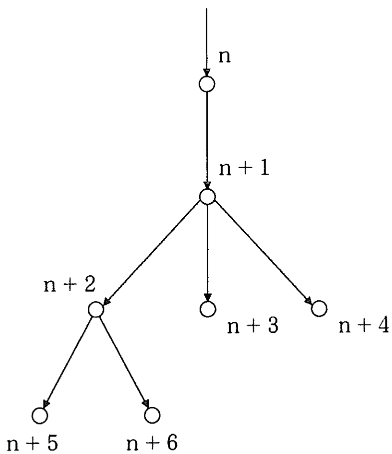

Figure 1 : An example of a directed tree

Let v be any vertex. Then, from the definition of the homogeneous m-th order Markov tree, we see that $\{X_w, w \in V_{v_1}\}, \cdots,$ and $\{X_w, w \in V_{v_{c(v)}}\}$ are conditionally independent given $(X_v, X_{v^1}, \cdots, X_{v^b})$, where $b=\min\{a(v), m-1\}$. We shall illustrate this fact with an example. For simplicity, we let $m=2$ and consider the part of the directed tree given in Figure 1. By using the definition of the homogeneous second order Markov tree, we have for every $e, e_1, e_2, \cdots, e_6$,

$$
\begin{aligned}
&P(X_{n+2}=e_2, X_{n+3}=e_3, X_{n+4}=e_4, X_{n+5}=e_5, X_{n+6}=e_6 \mid X_n=e, \\
&\quad X_{n+1}=e_1) \\
&=P(X_{n+2}=e_2 \mid X_n=e, X_{n+1}=e_1) \cdot P(X_{n+3}=e_3 \mid X_n=e, X_{n+1}=e_1, \\
&\quad X_{n+2}=e_2) \\
&\quad \cdot P(X_{n+4}=e_4 \mid X_n=e, X_{n+1}=e_1, X_{n+2}=e_2, X_{n+3}=e_3) \\
&\quad \cdot P(X_{n+5}=e_5 \mid X_n=e, X_{n+1}=e_1, X_{n+2}=e_2, X_{n+3}=e_3, X_{n+4}=e_4) \\
&\quad \cdot P(X_{n+6}=e_6 \mid X_n=e, X_{n+1}=e_1, X_{n+2}=e_2, X_{n+3}=e_3, X_{n+4}=e_4, \\
&\quad X_{n+5}=e_5) \\
&=P(X_{n+2}=e_2 \mid X_n=e, X_{n+1}=e_1) \cdot P(X_{n+3}=e_3 \mid X_n=e, X_{n+1}=e_1) \\
&\quad \cdot P(X_{n+4}=e_4 \mid X_n=e, X_{n+1}=e_1) \\
&\quad \cdot P(X_{n+5}=e_5 \mid (X_n=e), X_{n+1}=e_1, X_{n+2}=e_2) \\
&\quad \cdot P(X_{n+6}=e_6 \mid (X_n=e), X_{n+1}=e_1, X_{n+2}=e_2) \\
&=P(X_{n+2}=e_2, X_{n+5}=e_5, X_{n+6}=e_6 \mid X_n=e, X_{n+1}=e_1) \\
&\quad \cdot P(X_{n+3}=e_3 \mid X_n=e, X_{n+1}=e_1) \cdot P(X_{n+4}=e_4 \mid X_n-e, X_{n+1}=e_1).
\end{aligned}
$$

Let k and m be positive integers. Assume that $\{X_v, v \in V\}$ is a homogeneous m-th order Markov tree defined above. Let $\phi(t)$ be the *pgf* of the distribution of the number of non-overlapping "1"-runs of length k along the direction in $\{X_v, v \in V\}$. For every vertex v except for the root $v(1)$, every $\mathbf{e} \in E_m$ and $\ell=0, 1, \cdots, k-1$ we let $\phi(v, \mathbf{e}, \ell; t)$ be the *pgf* of the conditional distribution of number of non-overlapping "1"-runs of length k along the direction in $\{X_w, w \in V_v\}$ given that at the vertex $pa(v)$ a "1"-run of length ℓ is observed and the current outcomes until $pa(v)$ is $\mathbf{e}$. For $e=0, 1$ and

$\mathbf{e}=(e_1, e_2, \cdots, e_j)$ $(1 \leq j \leq m)$, we define

$$f(\mathbf{e}, e)=\begin{cases}(e_1, \cdots, e_j, e), & \text{if } 1 \leq j < m,\\ (e_2, \cdots, e_j, e), & \text{if } j=m.\end{cases}$$

Theorem 2. 1. Under the above assumptions, the *pgf*'s satisfy the following recurrence relations;

$$\phi(t)=\begin{cases} q\prod_{j=1}^{c(v(1))}\phi(v(1)_j, (0), 0; t) & \text{if } c(v(1))>0 \text{ and } k>1,\\ \quad +p\prod_{j=1}^{c(v(1))}\phi(v(1)_j, (1), 1; t) & \\ q\prod_{j=1}^{c(v(1))}\phi(v(1)_j, (0), 0; t) & \text{if } c(v(1))>0 \text{ and } k=1,\\ \quad +pt\prod_{j=1}^{c(v(1))}\phi(v(1)_j, (1), 0; t) & \\ q+pt & \text{if } c(v(1))=0 \text{ and } k=1,\\ 1 & \text{if } c(v(1))=0 \text{ and } k>1,\end{cases} \tag{2. 1}$$

for $v \neq v(1)$,

$$\phi(v, \mathbf{e}, \ell; t)=\begin{cases} q(\mathbf{e})\prod_{j=1}^{c(v)}\phi(v_j, f(\mathbf{e}, 0), 0; t) & \text{if } c(v)>0 \text{ and } \ell<k-1,\\ \quad +p(\mathbf{e})\prod_{j=1}^{c(v)}\phi(v_j, f(\mathbf{e}, 1), \ell+1; t) & \\ q(\mathbf{e})\prod_{j=1}^{c(v)}\phi(v_j, f(\mathbf{e}, 0), 0; t) & \text{if } c(v)>0 \text{ and } \ell=k-1,\\ \quad +p(\mathbf{e})t\prod_{j=1}^{c(v)}\phi(v_j, f(\mathbf{e}, 1), 0; t) & \\ q(\mathbf{e})+p(\mathbf{e})t & \text{if } c(v)>0 \text{ and } \ell=k-1,\\ 1 & \text{if } c(v)>0 \text{ and } \ell<k-1,\end{cases} \tag{2. 2}$$

where $q=1-q$ and $q(\mathbf{e})=1-p(\mathbf{e})$.

Proof. First, we assume that $c(v(1))>0$ and $k>1$. Then, from the definition of the homogeneous m-th order Markov tree, $X_{v(1)_1}, \cdots, X_{v(1)_{c(v(1))}}$ are conditionally independent given $X_{v(1)}$. Further, from the above example with Figure 1, we see that $\{X_w, w \in V_{v(1)_1}\}, \cdots,$ and $\{X_w, w \in V_{v(1)_{c(v(1))}}\}$ are conditionally independent given $X_{v(1)}$. Note that the number of "1"-runs of length k observed in $\{X_v, v \in V\}$ is the sum of the numbers of "1"-runs of length k observed in $\{X_w, w \in V_{v(1)_1}\}, \cdots,$ and $\{X_w, w \in V_{v(1)_{c(v(1))}}\}$. Therefore, the first equation of (2.1) holds. Next, in case of $k=1$, we can observe a "1"-run of length 1 at $v(1)$ the root. Hence, if $c(v(1))>0$ then we obtain the second equation of (2.1). If $c(v(1))=0$ then we obtain $\phi(t)=q+pt$. If $c(v(1))=0$ and $k>1$ then we can not observe "1"-runs of length k at $v(1)$ the root and we obtain $\phi(t)=1$. Thus, we have (2.1).

For any vertex v except for the root, we let $c(v)>0$ and $\ell<k-1$. Then, from the definition of the homogeneous m-th order Markov tree and the above example with Figure 1, we can see that $\{X_w, w \in V_{v_1}\}, \cdots,$ and $\{X_w, w \in V_{v_{c(v)}}\}$ are conditionally independent given $(X_v, X_{v^1}, \cdots, X_{v^b})$, where $b=\min\{a(v), m-1\}$. Since the number of "1"-runs observed in $\{X_w, w \in V_v\}$ is the sum of the numbers of "1"-runs observed in $\{X_w, w \in V_{v_1}\}, \cdots,$ and $\{X_w, w \in V_{v_{c(v)}}\}$, the first equation of (2.2) holds. In case of $\ell=k-1$, we can observe a "1"-run of length k at v (the root of V_v). Hence, if $c(v)>0$ then we obtain the second equation of (2.2). And if $c(v)=0$ then we obtain $\phi(v, \mathbf{e}, \ell; t)=q(\mathbf{e})+p(\mathbf{e})t$. If $c(v)=0$ and $\ell<k-1$ then we can not observe "1"-runs of length k at v (the root of the subtree) and we obtain $\phi(v, \mathbf{e}, \ell; t)=1$. Thus, we have (2.2). This completes the proof.

Remark 2.1. In fact, the boundary conditions of the recurrence relations in Theorem 2.1 are given at every leaf since $c(v)=0$ is observed at every leaf. However, by taking account of the length of the remaining subtree at every vertex, we can add another type of boundary conditions; for $\ell=0, \cdots, k-1$ and g_{T_v} (the maximum length of the paths in T_v), $\phi(v, \mathbf{e}, \ell; t)=1$ if $\ell+g_{T_v}<k$.

Remark 2. 2. Letting $k=1$ in Theorem 2. 1, we obtain the distribution of the number of "1" in the m-th order Markov tree. In particular, when $k=1$ and $p=p(\mathbf{e})$ (*iid* case), we see that $\phi(t)=(q+pt)^N$, where N is the cardinality of V. This is the usual binomial distribution.

Remark 2. 3. Letting $m=1$ in Theorem 2. 1, we obtain the distribution of the number of non-overlapping "1"-runs in the Markov tree.

Remark 2. 4. When $c(v)=0$ or 1 for every $v \in V$ in Theorem 2. 1, the directed tree reduces to a sequence of a finite length and hence we can obtain the distribution of the number of non-overlapping "1"-runs in the m-th order Markov chain.

3. Computational aspects

Let $\{X_v, v \in V\}$ be a collection of $\{0, 1\}$-valued random variables indexed by the vertices of a directed tree with a higher order Markov tree is given. In this section, we illustrate here how to count the number of non-overlapping "1"-runs of length k by using the following example.

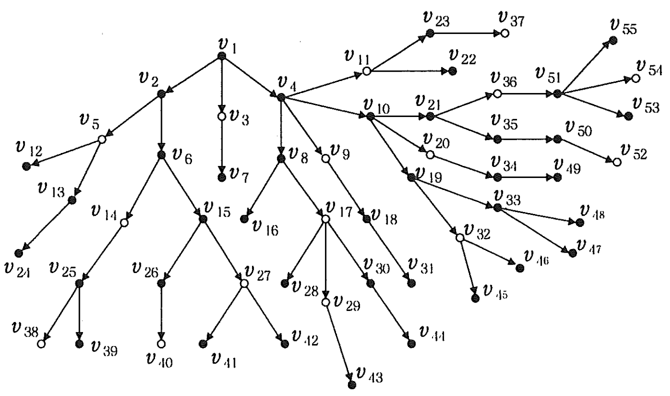

Figure 2 : A realization of a collection of binary random variables indexed by the vertex set of a directed tree

Example 3. 1. Figure 2 shows an example of a realization of a directed tree of binary ({●, ○}-valued) random variables. We illustrate how to enumerate the number of non-overlapping "●"-runs of length 3 on the directed tree of Figure 2. First, we adopt the non-overlapping counting method along the direction in every path from the root.

When a collection of {0, 1}-valued random variables indexed by the vertices of a directed tree with a higher order Markov tree is given, Theorems 2.1 indeed provide algorithms for deriving the *pgf*'s of the distributions of the numbers of "1"-runs of a specified length, respectively. We illustrate here how to derive the *pgf*'s by using the following example.

Example 3.2. We calculate the *pgf*'s of the distributions of the numbers of "1"-runs of length 3 on the directed tree of Figure 3. We assume that X_{v_j}, $j=1, \cdots, 20$ has the second order Markov tree. For the moment, we think the case of non-overlapping counting.

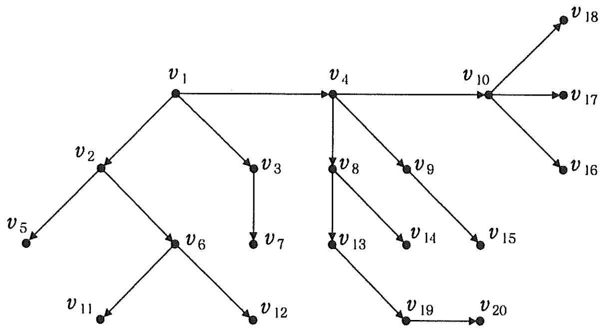

Figure 3 : An example of a directed tree

Since the root v_1 has the children $\{v_2, v_3, v_4\}$, we have from the first equation of (2.1)

$$\begin{aligned}\phi_{v_1}(t) = & q\phi(v_2, (0), 0; t)\phi(v_3, (0), 0; t)\phi(v_4, (0), 0; t) \\ & +p\phi(v_2, (1), 1; t)\phi(v_3, (1), 1; t)\phi(v_4, (1), 1; t).\end{aligned}$$

Next, noting that the vertex v_2 has children $\{v_5, v_6\}$, we have

$$\phi(v_2, (0), 0; t) = q(0)\phi(v_5, (0,0), 0; t)\phi(v_6, (0,0), 0; t) + p(0)\phi(v_5, (0,1), 1; t)\phi(v_6, (0,1), 1; t),$$
$$\phi(v_2, (1), 1; t) = q(1)\phi(v_5, (1,0), 0; t)\phi(v_6, (1,0), 0; t) + p(1)\phi(v_5, (1,1), 2; t)\phi(v_6, (1,1), 2; t).$$

Since the vertex v_5 does not have a child, from the last equation of (2.2), we have $\phi(v_5, (0,0), 0; t) = 1$, $\phi(v_5, (0,1), 1; t) = 1$ and $\phi(v_5, (1,0), 0; t) = 1$. Further, from the third equation of (2.2), we have $\phi(v_5, (1,1), 2; t) = q(1,1) + p(1,1)t$. From Remark 2.1, we observe $\phi(v_6, (0,0), 0; t) = 1$, $\phi(v_6, (1,0), 0; t) = 1$. Further, by using the all equations of (2.2), we have

$$\phi(v_6, (0,1), 1; t) = q(0,1) + p(0,1)\phi(v_{11}, (1,1), 2; t)\phi(v_{12}, (1,1), 2; t),$$
$$\phi(v_6, (1,1), 2; t) = q(1,1) + p(1,1)t.$$

Here, from the third equation of (2.2), we have

$$\phi(v_{11}, (1,1), 2; t) = q(1,1) + p(1,1)t,$$
$$\phi(v_{12}, (1,1), 2; t) = q(1,1) + p(1,1)t.$$

Consequently, we obtain

$$\phi(v_2, (0), 0; t) = q(0) + p(0)(q(0,1) + p(0,1)(q(1,1) + p(1,1)t)^2),$$
$$\phi(v_2, (1), 1; t) = q(1) + p(1)(q(1,1) + p(1,1)t)^2.$$

Similarly, we have

$$\phi(v_3, (0), 0; t) = 1,$$
$$\phi(v_3, (1), 1; t) = q(1) + p(1)(q(1,1) + p(1,1)t),$$
$$\begin{aligned}\phi(v_4, (0), 0; t) = q(0)\{&q(0,0)(q(0,0) + p(0,0)(q(0,1)\\ &+ p(0,1)(q(1,1) + p(1,1)t))) + p(0,0)(q(0,1)\\ &+ p(0,1)(q(1,1) + p(1,1)t))\}\\ &+ p(0)\{q(0,1)(q(1,0) + p(1,0)(q(0,1)\\ &+ p(0,1)(q(1,1) + p(1,1)t)))\\ &+ p(0,1)(q(1,1) + p(1,1)t)^2\}\end{aligned}$$

$$\times\{q(0,1)+p(0,1)(q(1,1)+p(1,1)t)\}$$

$$\times\{q(0,1)+p(0,1)(q(1,1)+p(1,1)t)^3\},$$

$$\begin{aligned}\phi(v_4,(1),1;t)=&q(1)\{q(1,0)(q(0,0)+p(0,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(1,0)(q(0,1)+p(0,1)(q(1,1)+p(1,1)t))\}\\&+p(1)\{q(1,1)(q(1,0)+p(1,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(1,1)t(q(1,1)+p(1,1)(q(1,1)\\&+p(1,1)(q(1,1)+p(1,1)t)))\}\\&\times\{q(1,1)+p(1,1)t\}^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\phi_{v_1}(t)=&q[q(0)+p(0)(q(0,1)+p(0,1)(q(1,1)+p(1,1)t)^2)]\\&\times[q(0)\{q(0,0)(q(0,0)+p(0,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(0,0)(q(0,1)+p(0,1)(q(1,1)+p(1,1)t))\}\\&+p(0)\{q(0,1)(q(1,0)+p(1,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(0,1)(q(1,1)+p(1,1)t)^2\}\times\{q(0,1)+p(0,1)(q(1,1)\\&+p(1,1)t)\}\times\{q(0,1)+p(0,1)(q(1,1)+p(1,1)t)^3\}]\\&+p[q(1)+p(1)((q(1,1)+p(1,1)t)^2)]\\&\times[q(1)+p(1)(q(1,1)+p(1,1)t)]\\&\times[q(1)\{q(1,0)(q(0,0)+p(0,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(1,0)(q(0,1)+p(0,1)(q(1,1)+p(1,1)t))\}\\&+p(1)\{q(1,1)(q(1,0)+p(1,0)(q(0,1)\\&+p(0,1)(q(1,1)+p(1,1)t)))\\&+p(1,1)t(q(1,1)+p(1,1)(q(1,1)\end{aligned}$$

$$+p(1,1)(q(1,1)+p(1,1)t)))\}$$
$$\times\{q(1,1)+p(1,1)t\}^2].$$

The *pgf*'s are polynomials with respect to t. For deriving the probability functions, we only take out the coefficients of t^j of the expanded *pgf*'s. Indeed, we calculate the probability function of the distribution of the number of non-overlapping "1"-runs of length 3 on the above second order Markov tree. Letting N be the number of non-overlapping "1"-runs of length 3, we obtain $P(N=0), \cdots, P(N=8)$ from $\phi_{v_1}(t)$. Since the probability functions are very long and may not be suitable for filling all the polynomial in this place, we give the probability functions in Han and Aki (1999).

Theorems 2.1 indeed provide algorithms for the corresponding computations. In fact, it is easy to convert the theorems to recursive procedures available in some computer algebra systems. Though we used in Example 3.2 the directed tree of very small length with only 20 vertices to illustrate how to derive the *pgf* of the second order Markov tree, the size of a directed tree is not a problem if we input the directed tree to a computer by using a standard method like adjacency list representation. Indeed, we can treat the corresponding computational results for the directed tree with 55 vertices of Figure 2. The *pgf* of the distribution of the number of non-overlapping "1"-runs of length 3 on the directed tree becomes a polynomial in t of degree 23, which is very long and may not be suitable for printing all the polynomials. Nevertheless, it is very easy to obtain the probabilities from the *pgf* by means of some computer algebra systems. In fact, most of the computer algebra systems are excellent in expanding polynomials and in taking out the coefficients.

First, we obtain the probability functions of the numbers of non-overlapping "1"-runs of length 3 along the direction on the -valued second order Markov tree by using the directed tree of Figure 2. By setting $p=0.5$, $p(0)=0.5$, $p(1)=0.6$, $p(0,0)=0.5$, $p(0,1)=0.6$, $p(1,0)=0.6$, $p(1,1)=0.7$, from Theorem 2.1, we have the probability functions of the numbers of non-overlapping "1"-runs of length 3 along the direction on the $\{0,1\}$-valued second order Markov tree with 55 vertices. Figure 4 shows the bar graph of the probability functions of the numbers of non-overlapping "1"-runs of length 3 along the direction on the $\{0,1\}$-valued second order Markov tree.

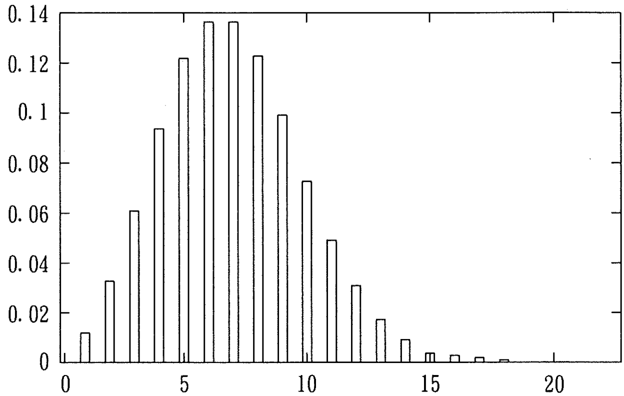

Figure 4 : the probability function of the number of non-overlapping "1"-runs of length 3 along the direction on a {0, 1}-valued second order Markov tree with 55 vertices

Next, we have the probability function of the number of non-overlapping "1"-runs of length 3 along the direction on the $\{0,1\}$-valued first order Markov tree by using the directed tree of Figure 2 when $p=0.8$, $p(0)=0.4$, $p(1)=0.8$. Figures 5 shows the bar graphs of the probability function of the number of non-overlapping "1"-runs of length 3 along the direction on the $\{0,1\}$-valued first order Markov tree. Further, we have the probability function of the number of non-overlapping "1"-runs of length 3 along the direction when the directed tree of Figure 2 is *iid* and $p=0.8$. Figures 6 shows the bar graphs of the probability function of the number of non-overlapping "1"-runs of length 3 along the direction when the $\{0,1\}$-valued directed tree is *iid.*

4. Conclusions

In the above approaches, we can see that the method of conditional probability generating functions is very useful for investigating the runs problems on a directed tree. Although the resulting system of equations of conditional probability generating functions is no longer linear and has complex structure, it is too easy and fast to obtain the probabilities from the algorithm of *pgf* by means of some computer algebra systems. The main purpose of this paper is to show how to use the method of conditional probability generating functions.

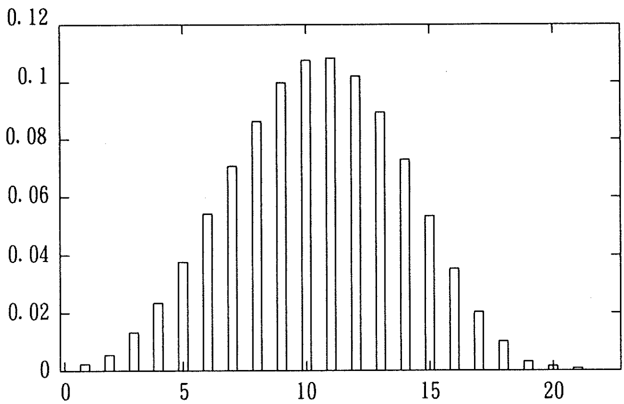

Figure 5 : the probability function of the number of non-overlapping "1"-runs of length 3 along the direction on a $\{0, 1\}$-valued first order Markov tree with 55 vertices

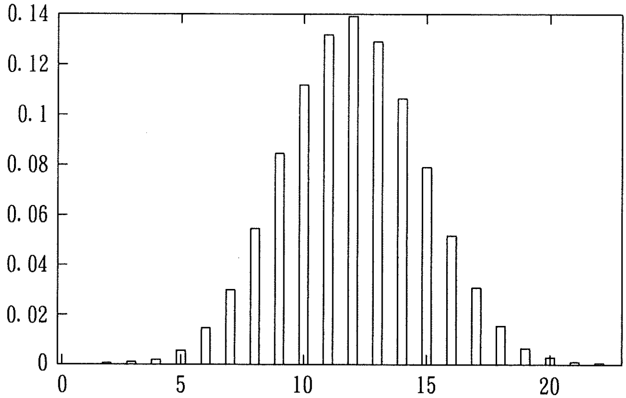

Figure 6 : the probability function of the number of non-overlapping "1"-runs of length 3 along the direction when the {0, 1}-valued directed tree with 55 vertices is *iid*

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