Piecewise Smooth Optimization Problems via Invex Methods and Applications concerning Walrasian Equilibrium

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Invexity was introduced as an extension of differentiable convex functions due to Hanson [6] in 1981. The idea plays an important role in analyzing various types of mathematical programming in which both feasible sets and objective functions are convex. For example, convex functions and affine functions are invex ones. In 1990 Karamardian et al [8] proved that generalized convexity of functions was equivalent to monotonicty of its gradient functions. It is said that the role in generalized monotonicty of the operator in variational inequality problems corresponding to the role in generalized convexity of objective functions in mathematical programming. Variational inequalities arise in models for a wide class of engineering or human sciences, e. g., mathematics, physics, economics, optimization and control, transportation, elasticity and applied sciences, etc. In this article we consider mathematical optimization problems and variational inequality problems. Finally we discuss the existence of Walrasian equilibrium of excess demand functions defined some feasible sets provided with invex properties.

Keywords: invex set, invex function, convex set, variational inequality problem, monotone function, Walrasian equilibrium

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1. Introduction

Consider the following mathematical problem

$\min f(x)$ subject to x in C, (MP)

where a feasible set C in $\boldsymbol{R}^n$ and an objective function f: $C \to \boldsymbol{R}$.

Here $\boldsymbol{R}$ and $\boldsymbol{R}^n$ are the set of real numbers, n-dimensional linear space, respectively. Problem (MP) is a particular case of the following variational inequality problems. In this paper we introduce an approach by applying the invex idea and to (MP) and the below problem variational inequality problems to x_0 in C satisfying

$(y-x_0)^T F(x_0) \geqq 0$ for y in C, (VIP)

where a function F: $C \to \boldsymbol{R}^n$ and x^T is the transpose of x. If f is differentiable and $F(x) = \nabla f(x)$, then (VIP) means (MP). According to the similar way as [9] we treat definitions of invexity in Section 2. Our aims are to solve variational-like inequality problems via the invex method (see Section 3) and to discuss invex feasible sets which are extended from the convex sets (see Section 4). In Section 5 we deal with applications to exchange price equilibrium. Finally, in Section 6 we give concluding remarks where we mention that it is possible to establish criteria for solutions to variational-like inequality problems corresponding to exchange price equilibrium.

2. Monotonicity and Invexity

In order to find optimal solutions for mathematical problems by finding solutions to variational inequality problems and those to variational-like inequality problems [9] discusses variationals of monotonicity and invexity.

Definition 1 A function $F: M \to \boldsymbol{R}^n$ is said to be *monotone*(M) on C if each x, y in C, then it follows that

$(y-x)^T(F(y)-F(x)) \geqq 0.$

A function F is said to be *pseudo monotone* (PM) on C if each x, y in C such that $(y-x)^T F(x) \geqq 0$, then $(y-x)^T F(y) \geqq 0$.

It follows that (M) means (PM) immediately. In [7] the following theorem is given as follows.

Theorem K A differential function f on an open set C is convex if and only if ∇f is monotone on C.

Definition 2 A function F is said to be invex *monotone* (IM) to a function $\eta: C^2 \to \boldsymbol{R}^n$ if for each x, y in C it follows that

$$\eta(y, x)^T[F(y)-F(x)] \geqq 0.$$

F is said to be *pseudo invex monotone* (PIM) to a function $\eta: C^2 \to \boldsymbol{R}^n$ if for each x, y in C with $\eta(y, x)^T F(x) \geqq 0$, then $\eta(y, x)^T F(y) \geqq 0$.

When F is (IM) to $\eta(y, x) = y-x$, it means that (IM) is (M). It follows that (IM) means (PIM).

The following examples illustrate (IM) and (PIM).

Example 1 Consider the following function $F(x) = x^2$ on $C = \{x \geqq 0\}$. It follows that F is (IM) to $\eta(y, x) = e^y - e^x$, since

$$\eta(y, x)[F(y)-F(x)]$$
$$=(y-x)(1+(y+x)/2+(y^2+yx+x^2)/3!+\cdots\cdots)[(y-x)(y+x)] \geqq 0.$$

Example 2 The following function

$$F(x) = -x \,(x<0);\ 0\,(x \geqq 0)$$

defined on $C = \boldsymbol{R}$ is not (IM) but (PIM) to the same $\eta(y, x) = e^y - e^x$. In case that $y < x \leqq 0$, we get $\eta(y, x)[F(y)-F(x)] = (e^y - e^x)(-y+x) < 0$, which means that F is un-(IM). If, however, $\eta(y, x)F(x) \geqq 0$, then $y \geqq x$ together with $\eta(y, x)F(y) \geqq 0$. Therefore F is (PIM) to the $\eta(y, x)$.

Definition 3 A Differentiable function f is said to be *invex* (IX) to a function $\eta: C^2 \to \boldsymbol{R}^n$ if, for each x, y in C, it follows that $f(y)-f(x) \geqq \eta(y, x)^T \nabla f(x)$.

Differentiable f is said to be pseudo invex (PIX) to a function η: $C^2 \to \boldsymbol{R}^n$ if, for each x, y in C with $\eta(y, x)^T f(x) \geqq 0$, it follows that $f(y) - f(x) \geqq 0$.

It follows that (IX) means (PIX). A function $f(x) = x + \sin x$ on $C = \{0 \leqq x < \pi/2\}$ is (IX) to $\eta(y, x) = (y + \sin y - x - \sin x)/(1 + \cos x)$, because

$$f(y) - f(x) = y + \sin y - (x + \sin x) = \eta(y, x) f'(x).$$

3. Variational-like Inequality Problems

In this section we treat variational-like inequality problems with respect to a parameter function η in order to find the following x_0 in C such that

$$\eta(y, x_0)^T F(x_0) \geqq 0 \text{ for } y \text{ in } C, \qquad \text{(VLIP)}$$

which plays an important role in solving optimal solutions for (MP) by utilizing the invex idea. We introduce definitions of hemi-continuity and invex sets. One means the continuity on linear segments and the other is an extension of convexity.

Definition 4 A function F is called hemi-continuous on C if for x, y in C, $y^T F(x + ty)$ is continuous on the closed interval $[0, 1]$ in $\boldsymbol{R}$.

Definition 5 A set M in $\boldsymbol{R}^n$ is an invex set to η: $C^2 \to \boldsymbol{R}^n$ if, for each x, y in C and t in $[0, 1]$, it follows that $x + t\eta(y, x)$ in C.

It can be easily seen that C is convex when C is invex to $y - x$. In the following example we show a different property of invex sets from that of convex sets.

Example 3 Let a subset M in $\boldsymbol{R}^2$ be invex to $\eta(y, x) = y$ on $C = \boldsymbol{R}^2 \times \boldsymbol{R}^2$. Denote vectors $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. Assume that $e_1, e_2 \in M$. Then we get

$$M = (\{1 \leqq x < \infty\} \times \boldsymbol{R}) \cup (\boldsymbol{R} \times \{1 \leqq y < \infty\}).$$

The following definition, lemma and theorem concerning KKM-functions play a significant role in guaranteeing the existence of optimal solutions of

(MP).

Definition 6 A function $V: \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ the power set of $\boldsymbol{R}^n$, is called *KKM-function* if, for every finite set $A=\{x_1, x_2, \cdots\cdots, x_m\}$ in $\boldsymbol{R}^n$, the convex hull conv(A) is contained in $\cup\{V(x_i): i=1, \cdots\cdots, m\}$.

Lemma 1 ([4]) Let a subset A in $\boldsymbol{R}^n$ be non-empty and $V: A \to 2^{\boldsymbol{R}^n}$ a KKM-function. If $V(x)$ is compact for x in A, then $\cap\{V(x): x \text{ in } A\} \neq \phi$.

Lemma 2 (Lemma 5.2 in [9]) Let C in $\boldsymbol{R}^n$ be non-empty, compact and convex. Let a function η be continuous, linear in the first argument and $\eta(x, y)+\eta(y, x)=0$ on C^2. If F is (PIM) to η and hemi-continuous on C, then the following statements (I)-(II) are equivalent each other.

(I) x in C satisfies (VLIP) to $\eta(y, x)$.

(II) x in C satisfies $\eta(y, x)^T F(y) \geqq 0$ for any y in C.

The above conditions of the parameter function η and the continuity of F are weakened as follows.

Lemma 3 (Extension of Lemma 2) Let E in $\boldsymbol{R}^n$ be non-empty, compact and convex. Let a function η be linear in the first argument and $\eta(x, x)=0$ for x in C. If F is (PIM) to η and η $(y, x)^T F(x)$ is upper-semicontinuous in x, then the following statements (I)-(II) are equivalent each other.

(I) x in C satisfies (VLIP) to $\eta(y, x)$.

(II) x in C satisfies $\eta(y, x)^T F(y) \geqq 0$ for any y in C.

Proof. (I) Let an x in C satisfy $\eta(y, x)^T F(x) \geqq 0$ for any y in C. Because of the (PIM) of F it follows that (II) holds,

(II) Let an x in C satisfy $\eta(y, x)^T F(y) \geqq 0$ for any y in C. Putting $w=x+t(y-x)$ for t in $[0, 1]$, from w in C, then we have $\eta(w, x)^T F(w) \geqq 0$. By the linearity of η and $\eta(x, x)=0$, we get

$$0 \leqq \eta(x+t(y-x), x)^T F(w)$$
$$=\eta(x, x)^T F(w)+t[\eta(y, x)^T F(w)-\eta(x, x)^T F(w)]$$

$= t\eta(y, x)^T F(w)$,

then $\eta(y, x)^T F(w) \geqq 0$ for $t>0$, This means that

$\limsup_{v\to x} \eta(y, x)^T F(v) \geqq \limsup_{w\to x} \eta(y, x)^T F(w) \geqq 0$. From the upper-semicontinuity, we have $\eta(y, x)^T F(x) \geqq \limsup_{v\to x} \eta(y, x)^T F(v) \geqq 0$. Therefore $\eta(y, x)^T F(y) \geqq 0$ means that $\eta(y, x)^T F(x) \geqq 0$. **Q. E. D.**

4. Existence Criteria for Invex Optimization Problems

In [9] authors gave the following existence criteria for invex problems.

Theorem R1 Assume that the same conditions of Lemma 2 hold. Then (VLIP) to η has at least one optimal solution in C.

In this studying we have an extension of the above criteria as follows.

Theorem 1 Assume that the same conditions of Lemma 3 hold. Then (VLIP) to η has at least one solution in C.

In order to prove the above theorem we prepare the following lemma.

Lemma 4 Let C in $\boldsymbol{R}^n$ be the same in Lemma 3. Then a subset

$V_1(y) = \{x \text{ in } C: \eta(y, x)^T F(x) \geqq 0\}$ (y in C) is a KKM-function.

Proof. Let $\{y_1, y_2, \cdots\cdots, y_n\}$ in C and $\sum_{i=1}^{n} a_i = 1$ with $a_i \geqq 0$ for $i=1, 2, \cdots\cdots, n$. Suppose that V_1 is not KKM, i. e., there exists some $y = \sum_{i=1}^{n} a_i y_i$ $a_i y_i$ in C such that y is not in $\cup\{V(y_i): i=1, \ldots n\}$ Then $\eta(y_i, y)^T F(y) < 0$ for any i and $\sum_{i=1}^{n} a_i \eta(y_i, y)^T F(y) < 0$. Since $\eta(x, y)$ is linear in x,

$0 = \eta(y, y)^T F(y) = \eta\left(\sum_{i=1}^{n} a_i y_i, y\right)^T F(y) = \sum_{i=1}^{n} a_i \eta(y_i, y)^T F(y) < 0$, which means a contradiction. Thus V_1 is a KKM-function. **Q. E. D.**

We shall prove Theorem 1.

Proof of Theorem 1. From Lemma 4, the function V_1 is a KKM-function. Since C is bounded, $V_1(y)$ is bounded for y in C. $V_1(y)$ is a closed subset in C, because $\eta(y, x)^T F(x)$ is upper-semicontinuous in x. Then $V_1(y)$ is compact in C for y. From Lemma 1, we have $\cap\{V_1(y): y \text{ in } C\} \neq \phi$, which means that (VLIP) has at least one solution in C. **Q. E. D.**

In [9] authors discussed the existence of optimal solution of (MP) under that the objective function f is differentiable and ∇f is (PIM). The latter means that f is (PIX). They mentioned that (PIX) is equivalent (IX) in case where f is an R-valued function. However, the proof of the equivalence between (PIX) and (IX) is not complete. In the following existence theorem for (MP) we give sufficient conditions that piecewise smooth f is (PIX) rather than (PIM) of ∇f and the upper semi-continuity of $\eta(y, x)^T F(x)$ in x.

Let a function f be piecewise smooth on an bounded open set C in $\boldsymbol{R}^n$. There exist at most finite points x_i in C, $i=1, 2, \cdots\cdots, n$, such that there exists no $\nabla f(x_i)$. Denote

$$\nabla f(y) = \limsup_{x \to y} \nabla f(x), \text{ where } f \text{ is not differentiable at } y. \qquad \text{(H)}$$

In what follows we assume that piecewise smooth function satisfies (H). Then it follows that f is upper semi-continuous on C.

Example 4 (i) Let $f(x) = |x|$ for x in R. When f satisfies (H), $f'(x) = -1\,(x<0)$; $f'(x) = 1\,(x \geqq 0)$. Thus f' is upper semi-continuous.

(ii) Let x in $C = [-4, 4]$ and let $f(x) = |x|$ for $-4 \leqq x < 0$; $f(x) = x$ for $0 \leqq x < 1$, $f(x) = x^{1/2}$ for $1 < x \leqq 4$. f is neither convex nor concave. Suppose that f satisfies (H). Then

$f'(x) = -1$ for $-4 \leqq x < 0$; $f'(x) = 1$ for $0 \leqq x \leqq 1$; $f(x) = 1/(2x^{1/2})$ for $1 < x \leqq 4$. Thus f' is upper semi-continuous.

Let C in $\boldsymbol{R}^n$ be an convex open set and f: $C \to \boldsymbol{R}^n$ be differentiable. In [9] they discussed the relation of between (PIM) of the gradient ∇f and (PIX) of f to the parameter function η: $C^2 \to \boldsymbol{R}^n$ with a positive condition that $\eta(y, x) > 0$ for x, y in C. We improve Theorem 4.7 in [9] and give a correct theorem.

Theorem 2 Let C in $\boldsymbol{R}^n$ be a non-empty convex open set and f: $C \to \boldsymbol{R}^n$ be differentiable. Assume that ∇f is (PIM) to $\eta(y, x) = k(y-x)$ for x, y in C, where k is a positive number. Then f is (PIX) to η.

Proof. Let $\eta(y, x)^T \nabla f(x) \geqq 0$ for x, y in C. Property (PIM) of ∇f means that $\eta(y, x)^T \nabla f(y) \geqq 0$ holds provided with $\eta(y, x)^T \nabla f(x) \geqq 0$. Putting $w = x + t(y-x)$ for t in $[0, 1]$ and x, y in C, we get $0 \leqq \eta(w, x)^T \nabla f(w) = kt(y-x)^T \nabla f(w)$ and $0 \leqq (y-x)^T \nabla f(x + t(y-x))$, which means that, by integrating over $[0, 1]$, $0 \leqq f(y) - f(x)$. Thus f is (PIX) to η. **Q. E. D.**

We give the following theorem which has improved conditions of Theorem 5.3 in [9].

Theorem 3 Let C in $\boldsymbol{R}^n$ be a non-empty convex compact set. Suppose that f is continuous and piecewise smooth and that f satisfies (H). Assume that ∇f is (PIM) to $\eta(y, x) = k(y-x)$ for x, y in C, where $k > 0$ and that $\eta(y, x)^T \nabla f(x)$ is upper semi-continuous in x. Then there exists at least one optimal solution of (MP) in C.

Proof. The parameter $\eta(y, x) = k(y-x)$ is linear in the first variable and satisfies $\eta(x, x) = 0$ for any x in C. From Theorem 1, (VLIP) to η and $F = \nabla f$ has at least one solution in C, i.e., an x in C satisfies $\eta(y,x)^T \nabla f(x) \geqq 0$. Because of (PIM) of ∇f to $\eta(y, x) = k(y-x)$, by Theorem 2, f is (PIX) to η. Then, by the definition of (PIX), the x satisfies $f(y) \geqq f(x)$ for y in C. Therefore (MP) has an optimal solution x. **Q. E. D.**

In order to treat the existence and uniqueness of optimal solution of (MP), we introduce the strict pseudo invexity to a parameter function.

Definition 7 Let C in $\boldsymbol{R}^n$ be an open set, f: $C \to \boldsymbol{R}^n$ be differentiable and η: $C^2 \to \boldsymbol{R}^n$. Function f is called *strictly pseudo invex* (SPIX) to η, if $f(y) > f(x)$ holds provided with $\eta(y, x)^T \nabla f(x) \geqq 0$ for $x \neq y$ in C. Function F: $C \to \boldsymbol{R}^n$ is called *strictly pseudo invex monotone* (SPIM) to η, if $\eta(y, x)^T F(y) > 0$ holds provided with $\eta(y, x)^T F(x) \geqq 0$ for $x \neq y$ in C.

In the following lemma we deal with the existence of solution for (VLIP) under conditions that F is (SPIM) and is not hemi-continuous but upper semi-continuous.

Theorem 4 (Extension of Corollary 5.1 in [9]) Let C in $\boldsymbol{R}^n$ be non-empty, compact and convex. Let a function η be linear in the first argument and $\eta(x, x)=0$ for x in C. If F is (SPIM) to η and $\eta(y, x)^T F(x)$ is upper semi-continuous in x. Then (VLIP) to η has at least one solution in C.

Proof. The above lemma can be proved in the similar way of the proof of Theorem 1. **Q. E. D.**

In the following theorem we get a criterion for the existence and uniqueness of optimal solutions for (MP).

Theorem 5 (Extension of Theorem 5.4 in [9]) Let C in $\boldsymbol{R}^n$ be a non-empty convex compact set. Suppose that f is continuous and piecewise smooth and that f satisfies (H). Assume that ∇f is (SPIM) to $\eta(y, x)=k(y-x)$ for x, y in C, where $k>0$ and that $\eta(y, x)^T \nabla f(x)$ is upper semi-continuous in x in C. Then there exists at least one optimal solution of (MP) in C.

Proof. The above Theorem can be proved in the similar way of the proof of Theorem 3. **Q. E. D.**

We illustrate the above criterion for the existence and uniqueness of optimal solution to (MP).

Example 5 Let $C=[-4, 4]$ be convex and compact in $\boldsymbol{R}$. Consider an $\boldsymbol{R}$-valued function

$$f(x)=|x| \text{ for } -4 \leqq x<0 ; f(x)=x \text{ for } 0 \leqq x<1 ; f(x)=x^{1/2} \text{ for } 1 \leqq x \leqq 4.$$

Then f is piecewise smooth and neither convex nor concave. Assume that f satisfies Property (H). Let $\eta(y, x)=k(y-x)$ with $k>0$.

(i) It follows that the gradient of f is as follows.

$f'(x)=-1$ for $-4\leqq x<0; f'(x)=1$ for $0\leqq x\leqq 1; f'(x)$

$=1/(2x^{1/2})$ for $1<x\leqq 4$.

Thus f' is upper semi-continuous.

(ii) If $(y-x)f'(x)\geqq 0$ holds, then $(y-x)f'(y)>0$. Thus f' is (SPIM).

(iii) If $(y-x)f'(x)\geqq 0$ holds, then $f(y)>f(x)$. Thus f is (SPIX).

(iv) The minimum of f over C is uniquely attained at $x=0$, and $\min f(x)=0$.

5. Applications Concerning Walrasian Equilibrium

In this section we treat an exchange economy with price vector $p=(p_1, p_2, \cdots\cdots, p_k)^T$ in the k-dimensional linear space $\boldsymbol{R}^k$ and induced aggregate excess demand function $z: D\rightarrow \boldsymbol{R}^k$. Let C be a subcone in the positive orthant $\boldsymbol{R}^k_+=\{p=(p_1, p_2, \cdots\cdots, p_k)^T: p_i\geqq 0,\ i=1, 2\cdots\cdots, k\}$ and $S^k_+=\left\{p \text{ in } \boldsymbol{R}^k_+: \sum_{i=1}^{k} p_i=1\right\}$. Denote the interior of $\boldsymbol{R}^k_+$ by $\boldsymbol{R}^k_{++}$. Let $D=S^k_+\cap C$. In usual $z(p)$ will be homogeneous of degree zero in p and will satisfy Walras's law, i. e., for $a>0$ and p in D, $z(ap)=z(p)$ holds; $p^Tz(p)=0$ on C. Moreover we assume that z is continuous on D. See [3] in details.

Example 6 Consider an excess demand of the Cobb-Douglas for two goods

$$z_1(p)=(am/p_1)-x_1,\ z_2(p)=(am/p_2)-x_2$$

where $p=(p_1, p_2)^T$, $m=p_1x_1+p_2x_2$ is money income, x_i is the demand for goods i for $i=1.2$, and a is a positive constant. The equality $p^Tz(p)=0$ means that $a=1/2$. Then $z(p)=0$ for any p.

Definition 8 A price vector p^* in D is called a *Walrasian equilibrium* if $z(p^*)\leqq 0$ in $\boldsymbol{R}^k$.

[3] showed equivalent conditions for the Walrasian equilibrium as follows.

Theorem 6 A price vector p^* in D is a Walrasian equilibrium satisfies the following conditions (a) and (b) each other.

(a) $z(p^*)^T(p-p^*) \leqq 0$ for p in S^k;

(b) $z(p^*)^T p \leqq 0$ for p in S^k.

Moreover assume that z is monotone on D, i. e.,

$$[z(p)-z(q)]T(p-q) \geqq 0 \text{ for } p, q \text{ in } D.$$

Then the above (a) and (b) are equivalent to the following conditions (c) and (d) mutually.

(c) $z(p)^T(p-p^*) \leqq 0$ for p in D.

(d) $z(p)^T p^* \geqq 0$ for p in D.

In [5] they get a sufficient condition for the existence of the Walrasian equilibrium.

Theorem 7 Let D be compact and convex and let $z: D \rightarrow \boldsymbol{R}^k$ be continuous. Then there exists at least one Walrasian equilibrium in D.

Finally, under weakened conditions that the set D is not necessarily compact but bounded and that z is not continuous but upper semi-continuous, we show some kind of possibility to the existence of a set of the Walrasian equilibriums.

Theorem 8 Let D be bounded and convex in $\boldsymbol{R}^k$. Let $-z$ be (PIM) to $\eta(q, p)=q-p$ and $\eta(q, p)^T(-z(p))$ lower semi-continuous in p for each q in D.

Then there exists at least one solution p^* for the following variational inequality problem

$$(q-p^*)^T z(p^*) \leqq 0 \text{ for any } q \text{ in } D.$$

Proof. Since D is convex and $-z$ is not monotone but (PIM) to η, by Theorem 1, there exists at least one solution p^* for the variational inequality problem $(q-p^*)^T(-z(p^*)) \geqq 0$ on D. Therefore p^* satisfies $(q-p^*)^T z(p^*) \leqq 0$ for any q in D. **Q. E. D.**

6. Concluding Remarks

In the previous section we get the existence criterion for variational inequality problems via methods of the invex idea under the conditions that given functions are not continuous but semi-continuous and that the feasible sets are not bounded. Our aim of this investigation is to find the Walrasian equilibrium in the set D where the aggregate excess demand function is defined. In order to show the existence of the equilibrium we need to prove that the conclusion of Theorem 9 holds true on the wider set S_k than D.

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